

Finite volume effects using lattice chiral perturbation theory

Buğra Borasoy (Technische Universität München)
Randy Lewis (University of Regina)
Daniel Mazur (University of Regina)

Outline

1. context
2. lattice regularization
3. the pion mass in a finite volume
4. the pion decay constant in a finite volume
5. the pion form factor in a finite volume

1. Context

- It is important to quantify the difference between finite volume lattice QCD simulations and the desired infinite volume limit.
- As the lightest particle, the pion plays a key role in volume effects and chiral perturbation theory is the natural tool. [Gasser and Leutwyler, *Nucl. Phys.* B307, 763 (1988)]
- There has been a lot of recent activity, examples include
 - M.F.L. Golterman, K.C. Leung, *Phys. Rev.* D56, 2 (1997)
 - M. Golterman, E. Pallante, *Nucl. Phys. (Proc. Suppl.)* 83, 250 (2000)
 - C.J.D. Lin, G. Martinelli, E. Pallante, C.T. Sachrajda, G. Villadoro, *Phys. Lett.* B553, (2003)
 - D. Becirevic & G. Villadoro, *Phys. Rev.* D69, 054010 (2004)
 - G. Colangelo & S. Durr, *Eur. Phys. J.* C33, 543 (2004)
 - G. Colangelo & C. Haefeli, *hep-lat/0403025*plus studies involving baryons and studies involving heavy mesons.
- Physical results don't depend on regularization scheme.
- Lattice regularization which is numerically convenient because...
 - ... loop integrals become finite sums
 - ... divergences would only appear as $a \rightarrow 0$; we'll use $a \neq 0$ but small enough to be numerically irrelevant.

2. Lattice regularization

Lewis and Ouimet, Phys. Rev. D64, 034005 (2001); Borasoy, Lewis and Ouimet, Phys. Rev. D65, 114023 (2002)

Chiral perturbation theory is an expansion in inverse powers of

$$\Lambda_\chi \sim m_\rho \sim 4\pi F_\pi \sim 4\pi F_K \sim 1 \text{ GeV}$$

A Euclidean chiral Lagrangian for pseudoscalar mesons is

$$\begin{aligned} \mathcal{L}_M &= \mathcal{L}_M^{(2)} + \mathcal{L}_M^{(4)} + \mathcal{L}_M^{(6)} + \dots \\ \mathcal{L}_M^{(2)} &= \frac{F^2}{4} \text{Tr} \left(\sum_\mu \nabla_\mu U^\dagger \nabla_\mu U - \chi^\dagger U - \chi U^\dagger \right) \\ \mathcal{L}_M^{(4)} &= -L_1 \left(\sum_\mu \text{Tr}(\nabla_\mu U^\dagger \nabla_\mu U) \right)^2 - L_2 \sum_{\mu,\nu} \text{Tr}(\nabla_\mu U^\dagger \nabla_\nu U) \text{Tr}(\nabla_\mu U^\dagger \nabla_\nu U) \\ &\quad - L_3 \sum_{\mu,\nu} \text{Tr}(\nabla_\mu U^\dagger \nabla_\mu U \nabla_\nu U^\dagger \nabla_\nu U) + L_4 \sum_\mu \text{Tr}(\nabla_\mu U^\dagger \nabla_\mu U) \text{Tr}\{\chi^\dagger U + \chi U^\dagger\} \\ &\quad + L_5 \sum_\mu \text{Tr}(\nabla_\mu U^\dagger \nabla_\mu U (\chi^\dagger U + U^\dagger \chi)) - L_6 (\text{Tr}(\chi^\dagger U + \chi U^\dagger))^2 \\ &\quad - L_7 (\text{Tr}(\chi^\dagger U - \chi U^\dagger))^2 - L_8 \text{Tr}(\chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger) \\ &\quad + i L_9 \sum_{\mu,\nu} \text{Tr}(F_{\mu\nu}^R \nabla_\mu U \nabla_\nu U^\dagger + F_{\mu\nu}^L \nabla_\mu U^\dagger \nabla_\nu U) - L_{10} \sum_{\mu,\nu} \text{Tr}(U^\dagger F_{\mu\nu}^R U F_{\mu\nu}^L) \end{aligned}$$

where the fields are

$$\begin{aligned} U(x) &= \exp \left[\frac{-i\lambda^a \pi^a(x)}{F} \right], & \chi &= 2B \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} + \dots \\ L_\mu(x) &= \exp[-ia\ell_\mu(x)], & R_\mu(x) &= \exp[-iar_\mu(x)] \end{aligned}$$

This Lagrangian is invariant under a local chiral transformation,

$$\begin{aligned} U(x) &\rightarrow g(x)U(x)h(x) \\ R_\mu(x) &\rightarrow g(x)R_\mu(x)g^\dagger(x + a_\mu) \\ L_\mu(x) &\rightarrow h^\dagger(x)L_\mu(x)h(x + a_\mu) \end{aligned}$$

if the covariant derivatives are defined appropriately . . .

. . . use the nearest-neighbour derivative in $\mathcal{L}_M^{(2)}$ to avoid extra (unphysical) states,

$$\nabla_\mu^{(+)}U(x) = \frac{1}{a} [R_\mu(x)U(x + a_\mu)L_\mu^\dagger(x) - U(x)]$$

. . . use a symmetric derivative in $\mathcal{L}_M^{(4)}$ to preserve parity,

$$\nabla_\mu^{(\pm)}U(x) = \frac{1}{2a} [R_\mu(x)U(x + a_\mu)L_\mu^\dagger(x) - R_\mu^\dagger(x - a_\mu)U(x - a_\mu)L_\mu(x - a_\mu)]$$

The full action contains the usual Lagrangian term plus a less familiar term which arises from the integration measure,

$$\begin{aligned} S_M &= a^4 \sum_x \mathcal{L}_M(x) - \frac{1}{2} \sum_x \text{Tr} \ln \left[\frac{2(1 - \cos \Phi(x))}{\Phi^2(x)} \right] \\ \text{where } \Phi(x) &= \frac{2}{F} \sum_{a=1}^8 t^a \pi^a(x) \quad \text{with } t_{bc}^a = -if_{abc} \end{aligned}$$

Example: meson masses at leading order

Neglecting isospin violation ($m_l \equiv m_u = m_d$), the lowest order pion, kaon and eta two point functions are

$$\Gamma_{MM} = - \left[x_M^2 + \frac{4}{a^2} \sum_\mu \sin^2 \left(\frac{aq_\mu}{2} \right) \right] \quad \text{where} \quad x_\pi = \sqrt{2Bm_l}, \quad x_K = \sqrt{B(m_l + m_s)}, \quad x_\eta = \sqrt{\frac{2}{3}B(m_l + 2m_s)}.$$

The meson masses are therefore

$$m_M = \frac{2}{a} \operatorname{arcsinh} \left(\frac{ax_M}{2} \right)$$

Notice the existence of a Gell-Mann–Okubo relation,

$$3\sinh^2 \left(\frac{am_\eta}{2} \right) = 4\sinh^2 \left(\frac{am_K}{2} \right) - \sinh^2 \left(\frac{am_\pi}{2} \right)$$

which reproduces the conventional relation as $a \rightarrow 0$.

Example: meson masses at NLO

Loop integration is from $-\pi/a$ to π/a .

By including all one-loop diagrams and $\mathcal{L}_M^{(4)}$ tree-level pieces, the two point functions to next-to-leading order are found to be

$$\Gamma_{MM} = -\frac{1}{Z_M^{(+)} Z_M^{(\pm)}} \left\{ X_M^2 + \frac{4Z_M^{(\pm)}}{a^2} \sum_\mu \sin^2 \left(\frac{aq_\mu}{2} \right) + \left(\frac{1 - Z_M^{(\pm)}}{a^2} \right) \sum_\mu \sin^2(aq_\mu) \right\}$$

where X_M , $Z_M^{(+)}$ and $Z_M^{(\pm)}$ contain five Lagrangian parameters, L_4 , L_5 , L_6 , L_7 , L_8 , and a single integral,

$$W_4(\epsilon) \equiv \int_0^\infty dx I_0^4(x) \exp \left[-x \left(4 + \frac{\epsilon}{2} \right) \right]$$

where $I_0(x)$ is a Bessel function.
For example, the kaon mass is

$$m_K = \frac{2}{a} \operatorname{arcsinh} \left(\frac{aX_K}{2} \right)$$

where

$$\begin{aligned} X_K^2 &= x_K^2 - \frac{8}{F^2} x_K^2 (x_\pi^2 + 2x_K^2) (L_4 - 2L_6) - \frac{8}{F^2} x_K^4 (L_5 - 2L_8) + \frac{7x_K^2}{24a^2 F^2} + \frac{x_K^2}{6a^2 F^2} W_4(a^2 x_\eta^2) \\ &\quad - \frac{3x_\pi^2 x_K^2}{64F^2} W_4(a^2 x_\pi^2) - \frac{3x_K^4}{32F^2} W_4(a^2 x_K^2) - \frac{x_K^2 x_\eta^2}{192F^2} W_4(a^2 x_\eta^2) \end{aligned}$$

Notice that the kaon mass vanishes in the chiral limit ($m_l = m_s = 0$), indicating that the theory does indeed have exact chiral symmetry even for $a \neq 0$.

Example: renormalization of Gasser-Leutwyler coefficients

As $a \rightarrow 0$, loop integrals diverge but the infinities can be absorbed into renormalized parameters. The meson masses and the scale dependences of the counterterms are in analytic agreement with dimensional regularization. For example,

$$\begin{aligned} L_4^r(1/a_2) - 2L_6^r(1/a_2) - [L_4^r(1/a_1) - 2L_6^r(1/a_1)] &= -\frac{1}{36(4\pi)^2} \ln\left(\frac{a_2}{a_1}\right), \\ L_5^r(1/a_2) - 2L_8^r(1/a_2) - [L_5^r(1/a_1) - 2L_8^r(1/a_1)] &= \frac{1}{6(4\pi)^2} \ln\left(\frac{a_2}{a_1}\right), \\ 3L_7^r(1/a_2) + L_8^r(1/a_2) - [3L_7^r(1/a_1) + L_8^r(1/a_1)] &= \frac{5}{48(4\pi)^2} \ln\left(\frac{a_2}{a_1}\right), \end{aligned}$$

for sufficiently small lattice spacings a_1 and a_2 .

Power divergences also arise in loop integrals, and they get absorbed into Lagrangian parameters during renormalization.

3. The pion mass in a finite volume

Consider $N_f = 2$ at one-loop order with isotropic spacing a .

The number of lattice sites in a spatial(temporal) direction is $N_s(N_t)$.

$$\Gamma = \frac{\text{p} \rightarrow \text{---}}{\text{---} + \text{X}} + \frac{\text{p} \rightarrow \text{---}}{\text{---} + \text{O}_{\pi^+, \pi^0}}$$

$$\begin{aligned}\Gamma &= \Gamma_{\text{LO}} + \Gamma_{\text{NLO}}^{(a)} + \Gamma_{\text{NLO}}^{(b)} \\ \Gamma_{\text{LO}} &= -x_\pi^2 - \frac{2}{a^2} \sum_\mu (1 - \cos ap_\mu) \\ \Gamma_{\text{NLO}}^{(a)} &= -\frac{2}{3a^4 F^2} - \frac{2x_\pi^4}{F^2}(l_3 + l_4) - l_4 \frac{x_\pi^2}{a^2 F^2} \sum_\mu (1 - \cos 2ap_\mu) \\ \Gamma_{\text{NLO}}^{(b)} &= \frac{1}{6N_s^3 N_t a^4 F^2} \sum_k \left(\frac{112 + 5a^2 x_\pi^2 - 20 \sum_\mu \cos ap_\mu - 20 \sum_\mu \cos ak_\mu + 12 \sum_\mu \cos ap_\mu \cos ak_\mu}{a^2 x_\pi^2 + 2 \sum_\mu (1 - \cos ak_\mu)} \right)\end{aligned}$$

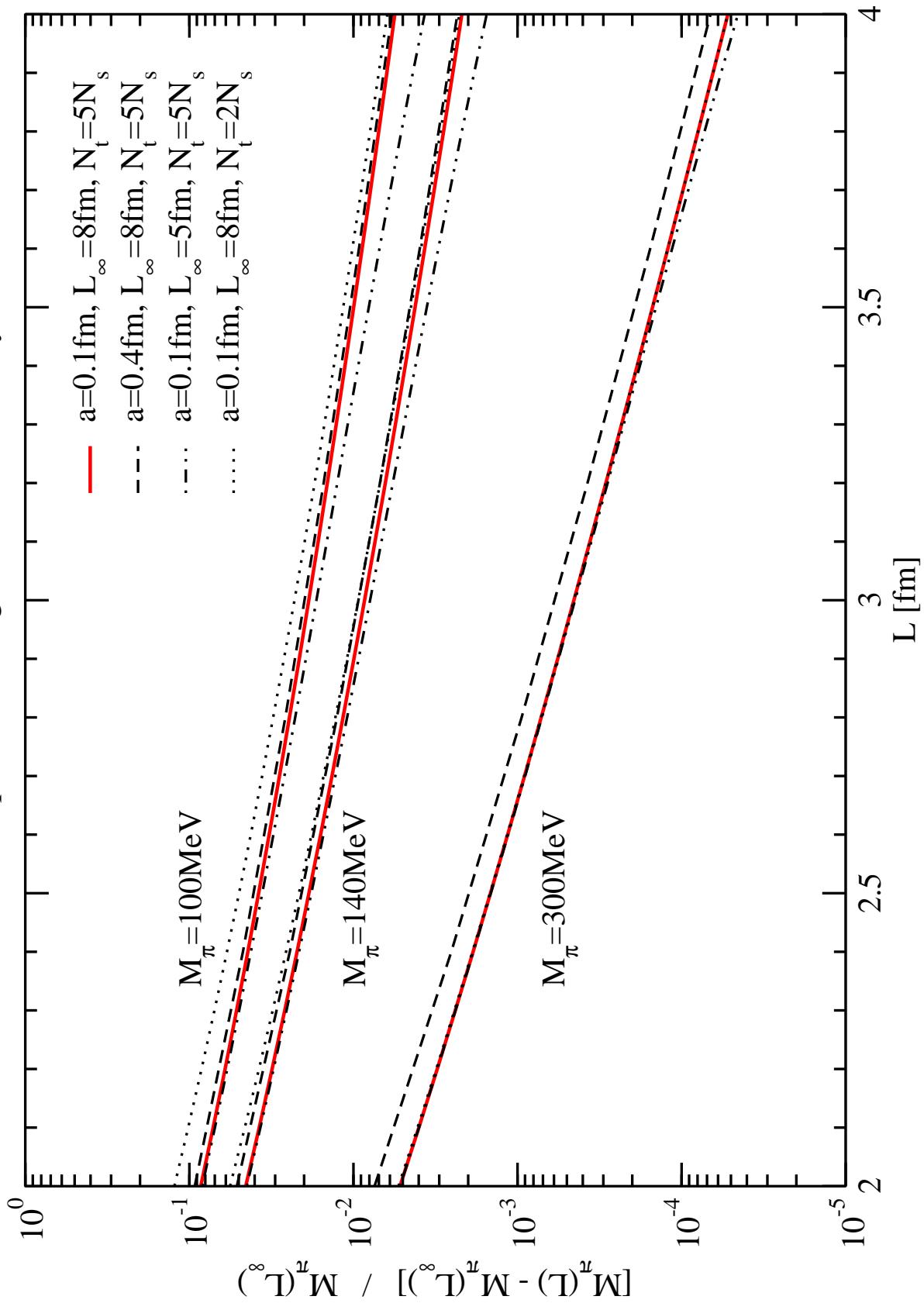
where the momentum summations include $k_{1,2,3} = \frac{2\pi n}{aN_s}$ with $n = 1, 2, 3, \dots, N_s$ and $k_4 = \frac{2\pi n}{aN_t}$ with $n = 1, 2, 3, \dots, N_t$.

The mass is obtained by solving $\Gamma = 0$ for $i\vec{p}_4$ with $\vec{p} = 0$. The result is

$$X_\pi^2 = x_\pi^2 + \frac{2x_\pi^4 l_3}{F^2} + \frac{x_\pi^2}{2N_s^3 N_t a^2 F^2} \sum_k \left(\frac{3 - 2 \cos ak_4}{a^2 x_\pi^2 + 2 \sum_\mu (1 - \cos ak_\mu)} \right) + O(a)$$

volume effects on the pion mass

value of F is taken from eq(30) of Colangelo & Dürr, Eur. Phys. J. C33, 543 (2004)



The expression for Γ provides wave function renormalization as well as mass renormalization. Since $N_s \neq N_t$, there is also an asymmetry parameter ξ_s , such that the propagator is

$$\frac{Z}{\xi(\tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2) + \tilde{p}_4^2 + X_\pi^2} \quad \text{where } \tilde{p}_\mu = 4 \sin^2 \left(\frac{p_\mu}{2} \right)$$

To determine Z , choose $\vec{p} = 0$ and define $\Sigma(-\tilde{p}_4^2)$ via

$$\Gamma = -[\tilde{p}_4^2 + x_\pi^2 + \Sigma(-\tilde{p}_4^2)]$$

This can be expanded as

$$\Gamma = -[\tilde{p}_4^2 + x_\pi^2 + \Sigma(X_\pi^2) + (-\tilde{p}_4^2 - X_\pi^2)\Sigma'(X_\pi^2) + \delta\Sigma(-p_4^2)]$$

where $\delta\Sigma(-p_4^2)$ vanishes like $(-\tilde{p}_4^2 - X_\pi^2)^2$ as $-\tilde{p}^2 \rightarrow X_\pi^2$. As usual then,

$$Z = \frac{1}{1 - \Sigma'(X_\pi^2)}$$

which immediately evaluates to

$$Z = 1 - \frac{2x_\pi^2 l_4}{F^2} + \frac{1}{3N_s^3 N_t a^2 F^2} \sum_k \left(\frac{5 - 3 \cos ak_4}{a^2 x_\pi^2 + 2 \sum_\mu (1 - \cos ak_\mu)} \right)$$

4. The pion decay constant in a finite volume

$$\Gamma = \left[\frac{p}{\sqrt{Z}} \right] \sqrt{Z} + \frac{p}{\sqrt{Z}} \times \text{wavy line} + \frac{p}{\sqrt{Z}} \times \text{loop}$$

For a stationary pion,

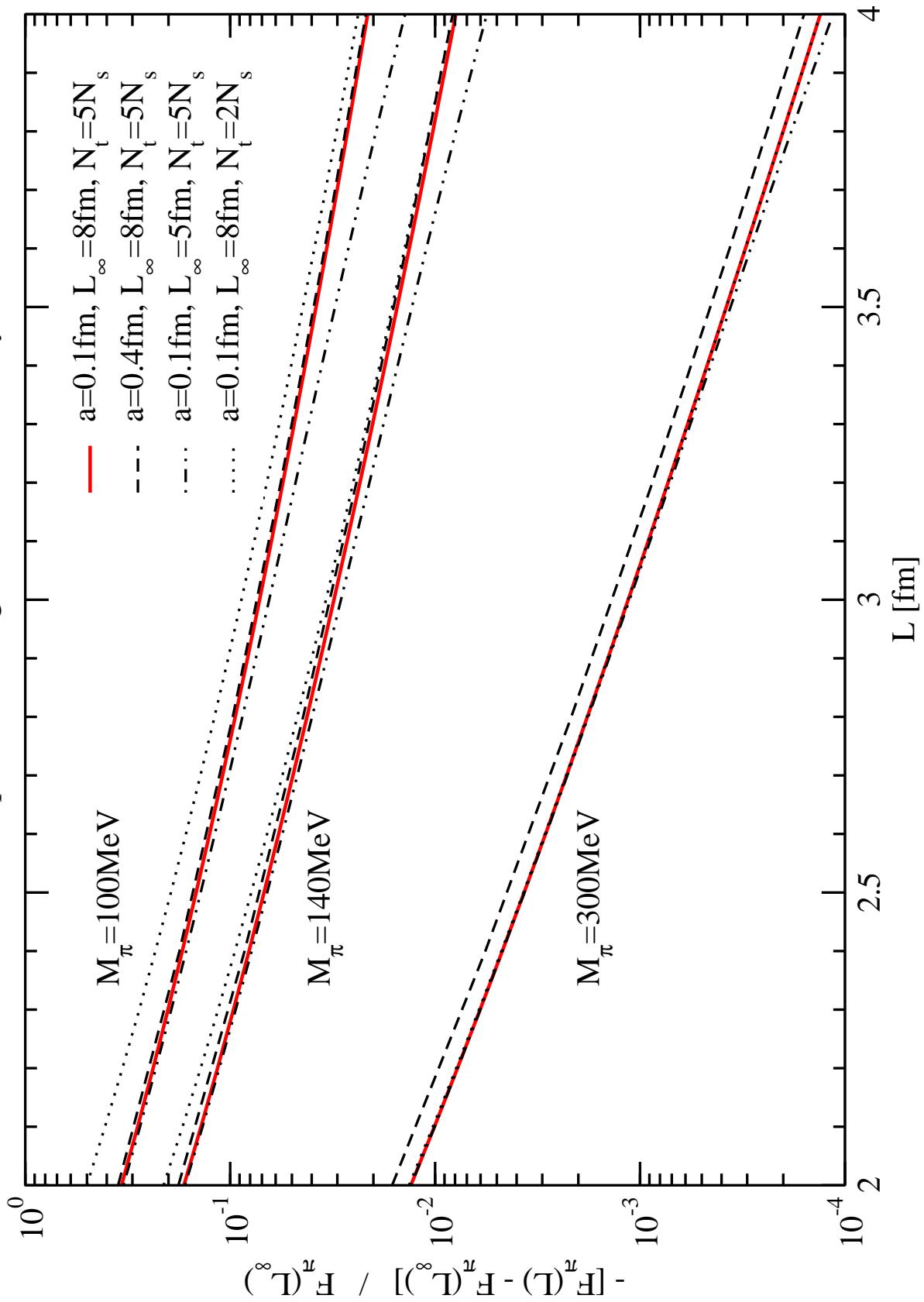
$$\begin{aligned}\Gamma &= \Gamma_{\text{LO}} \sqrt{Z} + \Gamma_{\text{NLO}}^{(a)} + \Gamma_{\text{NLO}}^{(b)} \\ \Gamma_{\text{LO}} &= \frac{i\sqrt{2}F}{a} \left[\sin ap_4 + 2i \sin^2 \left(\frac{ap_4}{2} \right) \right] \\ \Gamma_{\text{NLO}}^{(a)} &= \frac{i\sqrt{2}}{aF} x_\pi^2 l_4 \sin(2ap_4) \exp \left(\frac{iap_4}{2} \right) \\ \Gamma_{\text{NLO}}^{(b)} &= \frac{-i\sqrt{2}}{3a^3 N_s^3 N_t F} \left[\sin ap_4 + 2i \sin^2 \left(\frac{ap_4}{2} \right) \right] \sum_k \left(\frac{7 - 3 \cos ak_4}{a^2 x_\pi^2 + 2 \sum_\mu (1 - \cos ak_\mu)} \right)\end{aligned}$$

Summing, one obtains

$$F_\pi = F + \frac{x_\pi^2 l_4}{F} - \frac{1}{2N_s^3 N_t a^2 F} \sum_k \left(\frac{3 - \cos ak_4}{a^2 x_\pi^2 + 2 \sum_\mu (1 - \cos ak_\mu)} \right) + O(a)$$

volume effects on the pion decay constant

value of F is taken from eq(30) of Colangelo & Dürr, Eur. Phys. J. C33, 543 (2004)



5. The pion form factor in a finite volume

$$\Gamma = \boxed{\frac{p \rightarrow \sum q}{Z}} + \frac{p \rightarrow \sum q}{\pi^+, \pi^0} + \frac{p \rightarrow \sum q}{\pi^+ \pi^+}$$

In the Breit frame, on-shell pion energies cancel and the calculation is manifestly real.
Choose $\mu = 4$ for the photon.

$$\Gamma = \Gamma_{\text{LO}} Z + \Gamma_{\text{NLO}}^{(a)} + \Gamma_{\text{NLO}}^{(b)} + \Gamma_{\text{NLO}}^{(c)}$$

$$\Gamma_{\text{LO}} = \frac{2}{a} \exp(-iaq_4/2) \sin a(p+q/2)_4$$

$$\Gamma_{\text{NLO}}^{(a)} = \frac{2l_4 x_\pi^2}{a F^2} \exp(-iaq_4/2) \cos(aq_4/2) \sin a(2p+q)_4$$

$$+ \frac{2l_6}{a^3 F^2} \exp(-iaq_4/2) \sum_\nu [\sin ap_4 \sin a(p+q)_\nu - \sin a(p+q)_4 \sin ap_\nu] \sin aq_\nu \cos(aq_4/2)$$

$$\Gamma_{\text{NLO}}^{(b)} = \frac{-10 \exp(-iaq_4/2)}{3 N_s^3 N_t a^3 F^2} \sum_k \left(\frac{\sin a(p+q/2)_4}{a^2 x_\pi^2 + 2 \sum_\nu (1 - \cos ak_\nu)} \right)$$

$$\Gamma_{\text{NLO}}^{(c)} = \frac{4 \exp(-iaq_4/2)}{N_s^3 N_t a^3 F^2} \sum_k \left(\frac{\sin a(k+q/2)_4 \sum_\nu \cos a(k-p)_\nu}{[a^2 x_\pi^2 + 2 \sum_\nu (1 - \cos ak_\nu)][a^2 x_\pi^2 + 2 \sum_\nu (1 - \cos a(k+q)_\nu)]} \right)$$

The result is

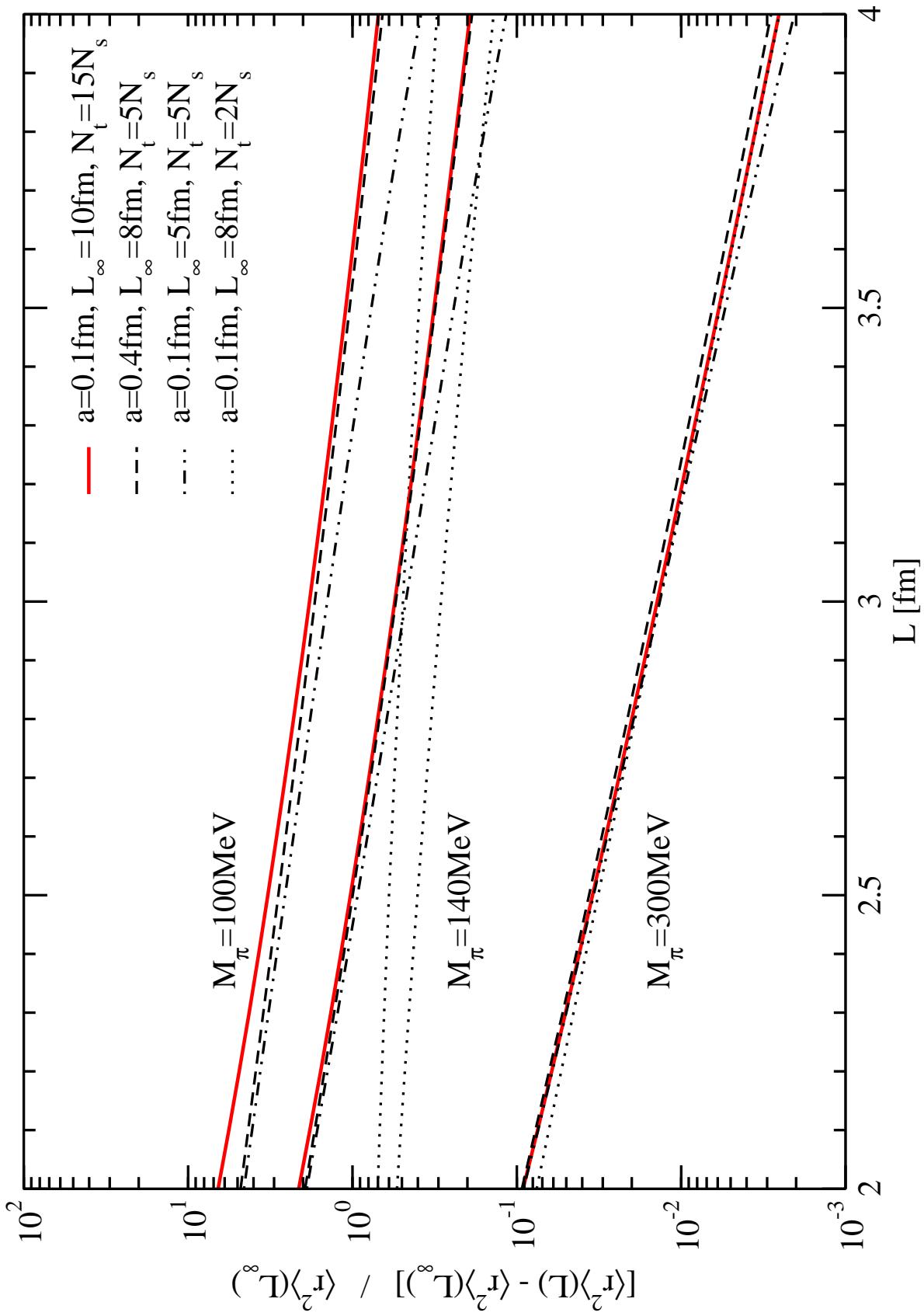
$$\begin{aligned} F_V(q^2) = & 1 + \frac{l_6}{a^2 F^2} \sin^2 a q_4 - \frac{1}{N_s^3 N_t a^2 F^2} \sum_k \left(\frac{\cos a k_4}{a^2 x_\pi^2 + 2 \sum_\mu (1 - \cos a k_\mu)} \right) \\ & + \frac{2}{N_s^3 N_t a^2 F^2} \sum_k \left(\frac{\sin^2 a k_4}{[a^2 x_\pi^2 + 2 \sum_\mu (1 - \cos a k_\mu)][a^2 x_\pi^2 + 2 \sum_\mu (1 - \cos a(k+q)_\mu)]} \right) + O(a) \end{aligned}$$

The charge radius is then obtained by differentiating.

$$\langle r^2 \rangle_V^\pi = \frac{-6l_6}{F^2} + \frac{12}{N_s^3 N_t F^2} \sum_k \frac{\sin^2 a k_4}{[a^2 x_\pi^2 + 2 \sum_\mu (1 - \cos a k_\mu)]^3} \left(\cos a k_1 - \frac{4 \sin^2 a k_1}{[a^2 x_\pi^2 + 2 \sum_\mu (1 - \cos a k_\mu)]} \right) + O(a)$$

volume effects on the pion charge radius

value of F is taken from eq(30) of Colangelo & Dürr, Eur. Phys. J. C33, 543 (2004)



Volume effects on the pion form factor

value of F is taken from eq(30) of Colangelo & Dürr, Eur. Phys. J. C33, 543 (2004)

